Extreme multistability in a Josephson-junction-based circuit

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We design and report an electrical circuit using a Josephson junction under periodic forcing that reveals extreme multistability. Its overall state equations surprisingly recall those of a well-known model of Josephson junction initially introduced in our circuit. The final circuit is characterized by the presence of two new and different current sources in parallel with the nonlinear internal current source $\sin[\phi(t)]$ of the Josephson junction single electronic component. Furthermore, the model presents an interesting extreme multistability which is justified by a very large number of different attractors (chaotic or not) when slightly changing the initial conditions.

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Multistability is characterized by a large number of coexisting attractors of many flavors, such as fixed points, limit cycles, and periodic or chaotic attractors, just for a given fixed set of parameters [1,2]. Many studies have addressed this phenomenon in a wide range of systems including electronic circuits [3], lasers [4], Josephson junctions (JJ) [5], biological systems [6], ecological systems [7,8], chemical reactions [9], and living neural systems [10]. The existence of a large number of attractors creates a complex structure in the landscape of basins of attraction and sometimes the systems may become extremely sensitive to small perturbations on the initial conditions. As a consequence, we can observe new qualitative behaviors. Also, small variations on the value of parameters sometimes cause dramatic changes due to change in the number of coexisting attractors, i.e., attractors appear and disappear quickly when a system parameter is varied [11]. This situation, for instance, may correspond to critical thresholds, such as tipping points [12], which are of great interest in the context of environment science, climate change, neuroscience, and social systems [13–15]. Here we address a circuit based on a Josephson-junction model to study multistability.

Cawthorne *et al.* proposed a Josephson-junction model that includes a resistive-capacitive-inductive shunting [16]. It consists of a resistive-capacitive-inductive shunted junction that has been used several times to produce chaotic dynamics [17,18]. In these specific references, the authors took advantage of its resistive-capacitive-inductive structure to obtain many interesting dynamical behaviors. Apart from these, Frolov *et al.* [19] developed a method to measure the current-phase relation of a π JJ, characterized by the minimum energy state being at a phase difference of π across the junction. Their technique was based on the relation $I_L = \phi(t)/L$, where L is the inductor in parallel to the superconductor-ferromagnet-superconductor Josephson junction and I_L is the current flowing through L (see Eqs. (3) and Fig. 2(a) in Ref. [19]). It can be noticed that the above expression of I_L is a particular solution of the equation $L\dot{I}_L(t) = \dot{\phi}(t)$, when the initial conditions of $I_L(t)$ and $\phi(t)$ respect the relation $\phi(0)/L - I_L(0) = 0$. What can happen if we assume that this relation between the initial conditions is not achieved, i.e., $\phi(0)/L - I_L(0) \neq 0$? This case is investigated in this article. To our surprise, we found that extreme multistable solutions exist as well as other interesting results, which will be described below.

The JJ is a device that has an internal state variable $\phi(t)$ expressed as $\dot{\phi}(t) = V_j$ where V_j is the voltage at points A and B of the circuit shown in Fig. 1. Thus, the dynamical stability of a chaotic circuit that contains a JJ could be closely dependent on the initial state of the JJ itself, as in the case of memristor-based chaotic circuits [20,21]. If that idea holds, then it may become possible to obtain the coexistence of infinitely many different attractors in the considered circuits. This gives rise to extreme multistability, where a tiny disturbance in the initial conditions may lead to different chaotic basins of attraction [22–27]. The existing literature already shows this phenomenon, which, to the best of our knowledge, has mostly been seen in coupled chaotic systems [22–27]. Within these works, we can recall Ref. [27], where the authors describe a method to construct self-reproducing systems for a unique class of systems. The method is based on a sliding of the variable that appear only as a linear term and the sliding constant is, in their own words, "an offset boosting controller for the variable; specifically the added constant can easily change the signal between unipolar and bipolar as desired for engineering applications since in many cases a specific physical circuit can only accept a unipolar

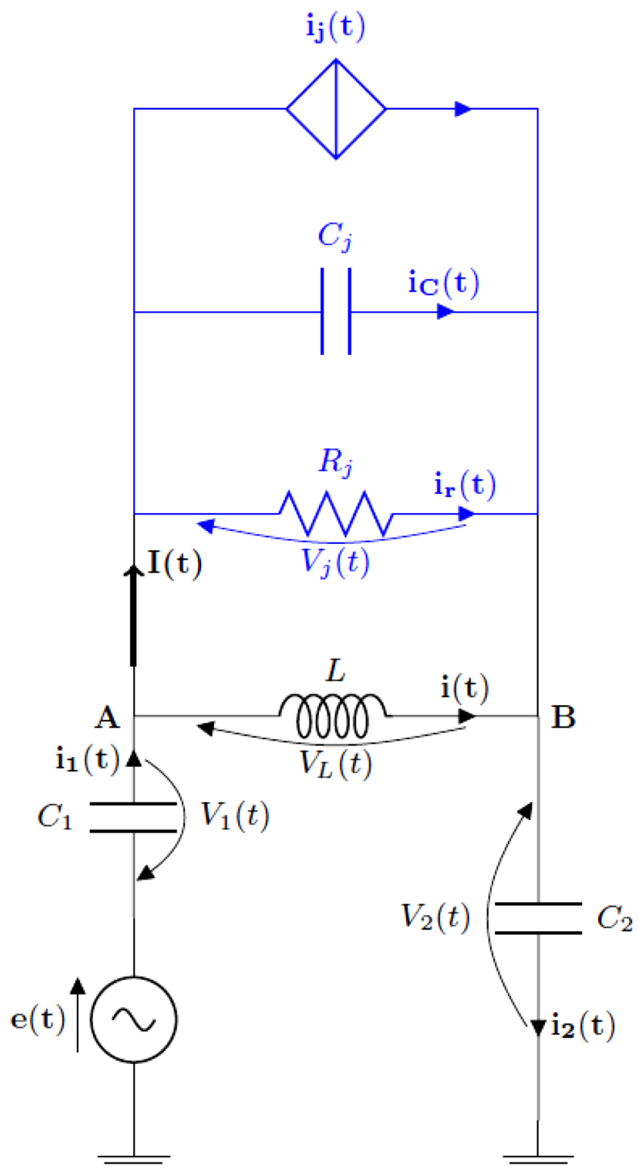

FIG. 1. Schematic diagram of the modified JJ extreme multistable circuit. This circuit is constituted of a Josephson junction element (blue part of the circuit [17,28–31]) connected in parallel to an induction L which is part of a nonautonomous C_1-L-C_2 resonator. The corresponding labels are defined in the text.

signal or a bipolar signal." (See p. 1750160-3, Ref. 27.) Thus, Ref. [27] seems to present the first single system that shows extreme multistability. Recently, Bao *et al.* [20] investigated the extreme multistability in an uncoupled system consisting of a Chua's circuit with a memristor-based nonlinearity. In this paper, we propose a system that includes a superconducting junction model which shows extreme multistability.

In the present work, we introduce an electric circuit with a Josephson junction under periodic forcing that reveals extreme multistability within sufficiently large parameter intervals suitable to possible experiments, as shown in Fig. 1. The rest of this work is organized as follows: Section 1 describes the mathematical modeling of the electric circuit and proposes an answer to the abovementioned question through some simulations. In Sec. II, the phenomenon of extreme multistability in the circuit is reported and justified, while Sec. III concludes the work.

I. THE CIRCUIT AND ITS MODEL

Figure 1 depicts the simple electrical circuit under study. A single-component JJ is connected in parallel to the inductor L, which is part of the C_1-L-C_2 tank circuit series with a periodic forcing, while the JJ, the nonlinear element, plays the feedback loop in this chaotic circuit.

Applying the Kirchhoff laws on Fig. 1, at nodes A and B,

$$i_1 = i_2 = I + i \quad \text{with} \quad I = i_j + i_C + i_r, \tag{1}$$

where $i_j(t) = \sin[\phi(t)]$. Furthermore, $i_1(t) = C_1\dot{V}_1(t)$, $i_c(t) = C_j\dot{V}_j(t)$, and $i_r(t) = \frac{V_j(t)}{R_j}$. From the loop formed by [$e(t)$, C_1, L, and C_2], the voltage equation is given as

$$e(t) - V_1(t) - V_L(t) - V_2(t) = 0 \quad \text{with} \quad V_L = V_j. \tag{2}$$

Without loss of generality, we consider $C_1 = C_2 = C$. This leads to $V_1(t) = V_2(t) = V(t)$, since $i_1(t) = i_2(t)$, and we can easily impose the initial condition of the capacitors C_i, $i = 1, 2$ to zero in practice. Therefore,

$$C\dot{V}(t) = I(t) + i(t). \tag{3}$$

The time derivative of the loop relation Eq. (2) gives $\dot{V}(t) = \frac{\dot{e}(t)-\dot{V}_j(t)}{2}$. Taking into account the node relations, Eq. (1),

$$I(t) = \frac{C}{2}[\dot{e}(t) - \dot{V}_j(t)] - i(t). \tag{4}$$

Then, from the JJ (blue part of the circuit in Fig. 1),

$$\dot{V}_j(t) = \frac{1}{C_j}\left[I(t) - \sin(\phi) - \frac{V_j(t)}{R_j}\right]. \tag{5}$$

The last two equations lead to

$$\dot{V}_j(t) = \frac{2C}{C + 2C_j}\left[\frac{\dot{e}(t)}{2} - \frac{1}{C}\sin(\phi) - \frac{V_j(t)}{R_jC} - \frac{1}{C}i(t)\right]. \tag{6}$$

Let us remember that $L\dot{i}(t) = V_L(t) = V_j(t)$ and $\dot{\phi}(t) = V_j(t)$. It implies that

$$i(t) = \frac{1}{L}\phi(t) + \frac{1}{L}\phi(0) - i(0). \tag{7}$$

Thus, it follows that

$$\dot{V}_j(t) = \frac{2C}{C + 2C_j}\left[\frac{\dot{e}(t)}{2} - \frac{1}{C}\sin(\phi) - \frac{V_j(t)}{R_jC} - \frac{1}{LC}\phi(t) - p\right], \tag{8}$$

where $p = \frac{1}{LC}\phi(0) - \frac{1}{C}i(0)$.

Finally, the dynamics of our system is described by the following set of first-order ordinary differential equations:

$$\dot{\phi}(t) = V_j(t), \tag{9a}$$

$$\dot{V}_j(t) = \frac{2C}{C + 2C_j}\left[\frac{\dot{e}(t)}{2} - \frac{1}{C}\sin(\phi) - \frac{V_j(t)}{R_jC} - \frac{1}{LC}\phi(t) - p\right]. \tag{9b}$$

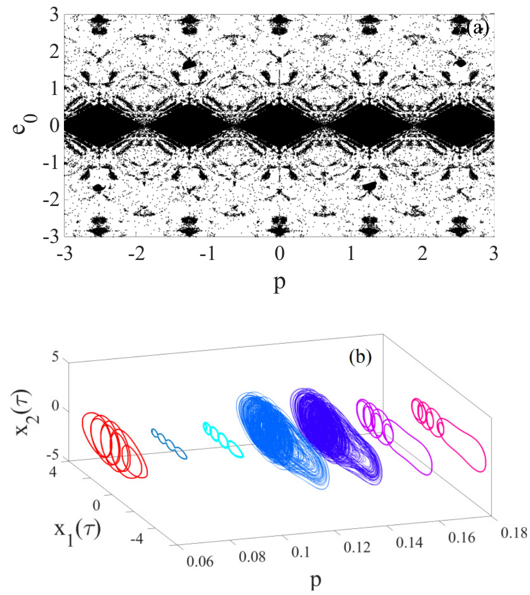

FIG. 2. (a) Domain of chaos (white dots) and regularity (black region) while varying simultaneously e_0 and the parameter p with a fixed value of $x_i(0) = 0$, $i = 1, 2$. This graph also shows that for $p = 0$ the system displays different behaviors when changing the value of e_0 for $a_1 = 0.21$, $a_2 = 0.001$, $a_3 = 0.2$, $w = 0.2$, $\beta = 10$, and $x_i = 0$, $i = 1, 2$. The parameters are fixed at $\beta = 10$ and $x_i(0) = 0$, $i = 1, 2$. As $x_1(0) = 0$, this particular case of $p = 0$ [meaning $i(0) = 0$] can be obtained by a discharge of the coil L. (b) Gallery of attractors as a function of p with $a_1 = 0.21$, $a_2 = 0.001$, $a_3 = 0.2$, $w = 0.2$, $e_0 = 0.51$, $\beta = 10$, and $x_i(0) = 0$, $i = 1, 2$.

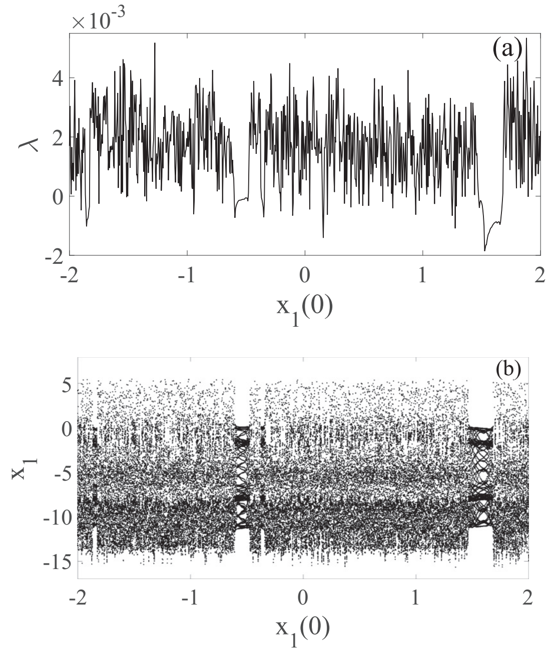

FIG. 3. Largest Lyapunov exponent and the corresponding bifurcation diagram obtained by varying $x_1(0)$ for a constant value of $p = 0$ in panels (a) and (b), All other parameters are kept constant at $x_2(0) = 0$, $e_0 = 1$, $w = 0.2$, $a_1 = 0.21$, $a_2 = 0.001$, $a_3 = 0.2$, and $\beta = 0.1$.

To facilitate the discussion, we define new variables: $x_1(t) = \phi(t)$, $x_2(t) = V_j(t)$, e_0, and w are respectively the amplitude and the frequency of $\frac{\dot{e}(t)}{2}$, $t = \Omega\tau$, and $\beta = \frac{2\Omega C}{C+2C_j}$, where Ω is a well-chosen constant, $a_1 = \frac{1}{C}$, $a_2 = \frac{1}{R_j C}$, and $a_3 = \frac{1}{LC}$, such that the circuit represented in Fig. 1 becomes dimensionless:

$$\dot{x}_1(\tau) = x_2(\tau), \quad \dot{x}_2(\tau) = \beta\{e_0 \sin(w\tau) - a_1 \sin[x_1(\tau)] - a_2 x_2(\tau) - a_3 x_1(\tau)\} - \beta p. \tag{10}$$

To look at the influence of parameter $p = \phi(0)/L_j - i(0)$, we first fix the initial condition $x_i(0) = 0$, $i = 1, 2$. [Let us recall that $x_1(t) = \phi(t)$ and $x_2(t) = V_j(t)$ are the only state variables. Thus, if $x_i(0)$, $i = 1, 2$ are fixed to zero, varying p implies varying $i(0)$ and that leads us to a sort of bifurcation diagram such as in Fig. 2(b), where the dynamics, as shown by the attractors, changes from chaos to regularity or from high to low amplitudes.] Figure 2(a) is a 2D projection of the largest Lyapunov exponent that gives beautiful structured domains of chaos and of regularity. The sign of the Lyapunov exponent (negative and positive) is represented by black and white respectively. This graph, obtained by varying the parameters e_0 and $i(0)$ for fixed $x_i(0) = 0$, $i = 1, 2$ exhibits a symmetry with respect to the origin (0,0). Having the possibility of varying the initial conditions of the external inductor L could be of great advantage in applications such as chaos-based secure communication [32,33], since the external initial condition $i(0)$ can be used as the information that has to be securely encrypted. Figure 2(b) gives a gallery of different attractors obtained as a function of p [meaning, $i(0)$, the charge of the inductor while we set in motion the circuit] with $a_1 = 0.21$, $a_2 = 0.001$, $a_3 = 0.2$, $w = 0.2$, $e_0 = 0.51$, $\beta = 10$, and $x_i = 0$, $i = 1, 2$. As $x_1(0) = 0$, the particular case of $p = 0$ [meaning $i(0) = 0$] can be obtained by a discharge of the coil L. Even if these graphs on Fig. 2 are sort of typical bifurcation diagram according to p, they help to answer the question related to p by showing that it could modify the measurement of the current-phase relation of a π JJ [19].

II. EXTREME MULTISTABILITY

Changing initial conditions in our model leads to different dynamics and to the coexistence of many attractors. This phenomenon defines the extreme multistability behavior [22–24]. Considering p, given as $p = \frac{1}{LC}\phi(0) - \frac{1}{C}i(0)$, different from zero could be questionable since such integration constant usually appears in the solution as an emergent property and it is not common in practice to add them in a model when anyone writes the equations using Kirchoff's law in a circuit. In this section, we investigate whether the system given by Eq. (10) is showing extreme multistability when $p = 0$. Keeping $\beta = 0.1$, $x_2(0) = 0$, $e_0 = 1$, $w = 0.2$, $a_1 = 0.21$, $a_2 = 0.001$, and $a_3 = 0.2$ shows us that varying the initial condition $x_1(0)$ (see Fig. 3) and $x_2(0)$ (see Fig. 4) implies changing the dynamics of the system, Eq. (10).

In this work, we have shown that a single electrical circuit using a Josephson junction model, for which a mathematical model is given by Eq. (10), suggests naturally without any transformation the presence of extreme multistability. The

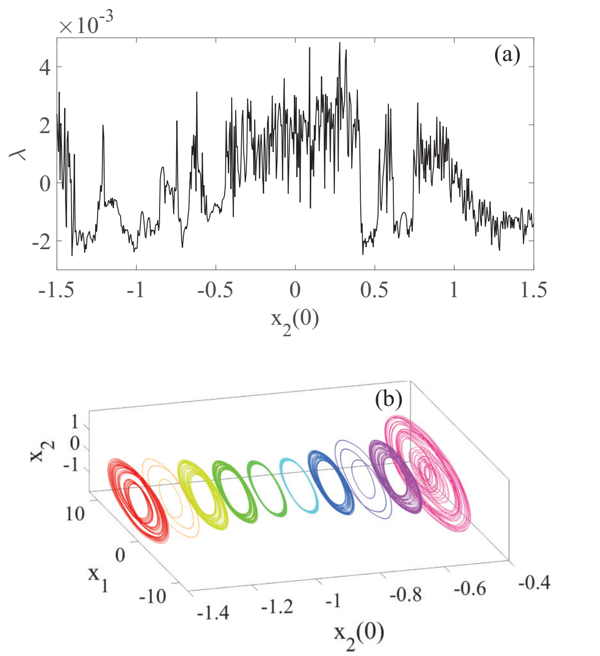

FIG. 4. Largest Lyapunov exponent and the corresponding bifurcation diagram obtained by varying $x_2(0)$ for a constant value of $p = 0$ in panels (a) and (b), All other parameters are kept constant at $x_1(0) = 0$, $e_0 = 1$, $w = 0.2$, $a_1 = 0.21$, $a_2 = 0.001$, $a_3 = 0.2$, and $\beta = 0.1$.

graph on Fig. 5 shows the plot of the sign of the largest Lyapunov exponent, represented in black and white, as a function of the initial conditions $x_1(0)$ and $x_2(0)$ in order to show the distribution of regions of chaos (black) and regularity (white) [34]. This graph reveals the intermingling basin that usually represents the extreme multistability. For this result, the unchanged parameters are the same as in Figs. 4 and 3.

III. CONCLUSION

We have designed a simple Josephson-junction-based electrical circuit that shows an interesting extreme multistability dynamics shown by Fig. 4(b). This conclusion is strengthened by the plot of Fig. 5 since this graph reveals the intermingling basin that usually represents the extreme multistability. The

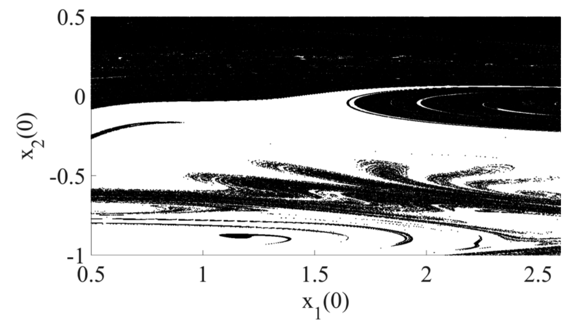

FIG. 5. Plot of the sign of the largest Lyapunov exponent, represented in black and white, as a function of the initial conditions $x_1(0)$ and $x_2(0)$ in order to show the distribution of regions of chaos (black) and regularity (white). For this result, the unchanged parameters are the same as in Figs. 4 and 3.

circuit we are dealing with is a phase-sensitive dissipative device, as suggested by Eq. (10), whose dependence on initial conditions reminds us of the behavior in systems with conservation laws, such as in ensembles of two-level atoms with a coherent feedback [35]. This behavior is illustrated by galleries of attractors given in Fig. 3 and supported by the bifurcation diagram and Lyapunov exponent depicted in Figs. 3 and 5. Thus, the answer to the question posed in the introduction of this article is now clear: It is possible that the initial charge of the coil influences the measurement of the current-phase relation of a π JJ [19]. However, we think that further investigations are needed for a better exploration and understanding of other aspects that this model may still be hiding.

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