Eur. Phys. J. B **23**, 97–102 (2001)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Coupling induced topological transition in a ring of chaotic oscillators

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Received 15 March 2001

Abstract. A ring of diffusively coupled Rössler oscillators, which can develop the conventional rotating wave from high-dimensional chaos by increasing the coupling ε continuously is studied. The chaotic generator for the rotating wave emerges around $\varepsilon = \varepsilon_0$, where the topological transition induced by the coupling not only changes the topological structure of all the oscillators, which share a common strange attractor, but also changes them into being different from each other. Starting from this transition, infinitely long range temporal correlation and spatial order in the style of antiphase state are established gradually, which gives rise to the chaotic generator of the rotating wave.

PACS. 05.45.Xt Synchronization; coupled oscillators – 05.45.Jn High-dimensional chaos

Since Turing analyzed rings of cells as models of morphogenesis and proposed that isolated rings could account for the tentacles of hydra and whorls of leaves of certain plants in 1952 [1], rings of coupled oscillators have been used extensively in physiological and biochemical studies [2,3], coupled laser systems, Josephson junction arrays, electrical circuits, coupled chemical oscillators, etc. [4–6]. There are three common types of couplings between elements of the rings which determine its symmetry. The first one is global coupling, *i.e.*, all to all coupling, mean-field coupling or star configuration (we regarded it as a ring for convenience). Each oscillator is identically coupled to all others. All permutations of n objects are involved, and we name its symmetry group S_n. The second is two-way coupling into a chain, most of which are diffusive coupling. Each oscillator is identically coupled to its nearest neighbor on both sides, and we name its symmetry group D_n. If one direction is preferred, we get the third case: one way coupling into a chain. We call its symmetry group, Z_n.

Golubitsky *et al.* [7] developed a general group-theoretical approach to study rings of coupled oscillators when temporal symmetry (phase shift between various oscillators) is taken into account, they implied that there may be a common stable dynamical state for the three types of coupling systems mentioned above. This state is the periodic rotating wave, in which each element evolves in the same waveform but $\frac{1}{n}$th of a period out of phase with each other. Since $Z_n \subset D_n$ and Z_n is a subgroup of D_n, D_n rotating-wave solutions can have two distinct directions of rotation, while only one sense of rotation will occur for Z_n. As for S_n, they occur with extraordinary multiplicity: the existence of one such attractor implies simultaneous coexistence of $(n-1)!$ such like attractors, since all permutations are guaranteed.

In early studies, interest was focused on coupled oscillators, each of which is periodic without coupling. This dynamical state had been one of the patterns of horse gait, centipedes crawling, lamprey swimming, etc. [3]. During the last decade, interest turned to the study of arrays of Josephson junction or multi mode lasers with S_n symmetry [4], in which the single oscillator can bifurcate into chaos and the chaotic behavior of the arrays has also been known. The typical rotating wave has been observed in these systems, and it was given several different names: antiphase solution, splay state, or ponies on a merry-go-round [4–6]. Since this dynamical state comes mainly from the symmetry of systems, we may expect it to be observed in coupled chaotic systems (*i.e.*, individual systems are chaotic without coupling). Recently, Matias *et al.* [8] reported the observation of a periodic rotating wave in rings of unidirectionally coupled analog chaotic oscillators, which are Z_n symmetric systems. As for D_n symmetric systems, Hu *et al.* [9] presented a periodic rotating wave in their study of the development of spatiotemporal chaos in diffusively coupled systems, with the name of antiphase state. We will use the latter in this context.

From the above, it seems that we can expect the rotating wave state, (*i.e.*, the antiphase state, the ponies on a merry-go-round), in all the three types of coupled chaotic oscillators. The remaining question is: how does the spatial order emerge from our high-dimensional chaotic sea? In the systems of periodic coupled oscillators comes to our help the symmetric Hopf bifurcation theorem of Golubitsky and Stewart [7], which is inseparable from the

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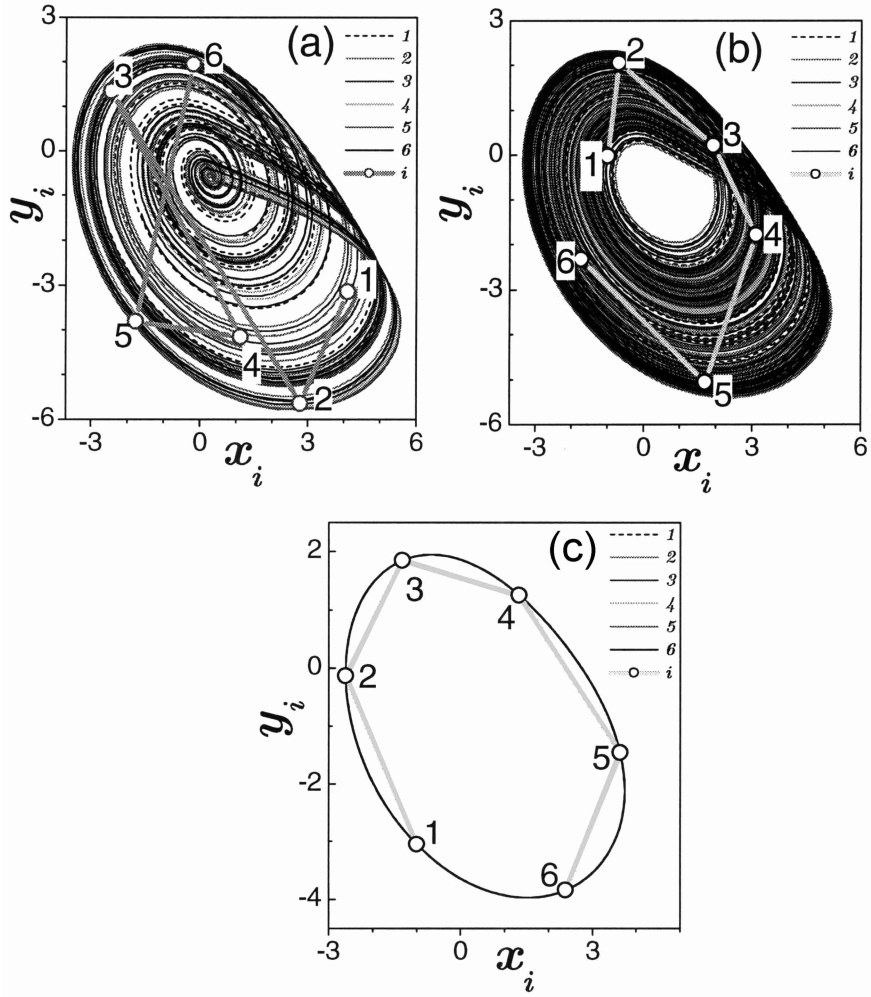

Fig. 1. Orbits of the coupling system in (x, y) space. (a) High-dimensional chaos without order either in time or spatial, $\varepsilon = 0.005$. (b) High-dimensional chaos with roughly spatial order, which can be regarded as a chaotic long time averaged-antiphase state, $\varepsilon = 0.018$. (c) The conventional periodic rotating wave, $\varepsilon = 0.080$. The numbers $i = 1, 2, ..., 6$ in the figures indicate the positions of the i oscillator at an arbitrary instant.

conceptions of equivariance and invariance under actions of symmetry. Regretfully it can not be applied as is under high-dimensional chaotic conditions with neither strict temporal nor strict spatial symmetry. As far as the systems of coupled chaotic oscillators are concerned, we have no fitting tools to resort, yet. One way to obtain the rotating wave is implied in reference [9] for a ring of coupled Rössler oscillators. That is by increasing the coupling strength continuously from zero. The system with D_n symmetry [9] can be written as

$$\begin{aligned} \dot{x}_i &= -y_i - z_i + \varepsilon\,(x_{i+1} + x_{i-1} - 2x_i) \\ \dot{y}_i &= \ x_i + a y_i + \varepsilon\,(y_{i+1} + y_{i-1} - 2y_i) \\ \dot{z}_i &= b + (x_i - c) z_i + \varepsilon\,(z_{i+1} + z_{i-1} - 2z_i) \\ x_{i+n} &= x_i,\ y_{i+n} = y_i,\ z_{i+n} = z_i; \quad i = 1, 2, ..., n. \end{aligned} \tag{1}$$

For $a = 0.45$, $b = 2.0$, and $c = 4.0$, the single Rössler oscillator is chaotic. We fix the system size to $n = 6$, so this is an 18-dimensional system. With the coupling $\varepsilon = 0$ and random initial conditions, all oscillators perform chaotic motions as the single oscillator does, however, they have asynchronous trajectories. This motion, which is random in both space and time, is shown in Figure 1a. When ε is increased larger than $\varepsilon_0 \approx 0.01$, averaged spatial order will emerge and form a chaotic generator for the conventional rotating wave. Figure 1b gives an example of this kind of chaotic averaged-antiphase state when $\varepsilon = 0.018$, and the numbers $i = 1, 2, ...n$ in the figure indicate the position of the ith oscillator at an arbitrary instant. Figure 1b shows that each oscillator is roughly $(\frac{2\pi}{n})$ out of phase (defined as the angle of polar coordinate in (x, y) space) with its neighbor. The averaged (or say, roughly) spatial symmetry is transferred by the chaotic generator of averaged-antiphase state, and will be transferred along the process of increasing coupling strength. When $\varepsilon > 0.057$, the system finally develops into a periodic rotating wave as shown in Figure 1c, in which each oscillator evolves in one common limit cycle but $(\frac{2\pi}{n})$ out of phase with each other. From spatially random chaos to a periodic rotating wave, the system should change from spatial disorder to ordered arrangement, and establish infinitely long range temporal correlations. Among the whole variation process, the transition in the vicinity around $\varepsilon_0 \approx 0.010$ is

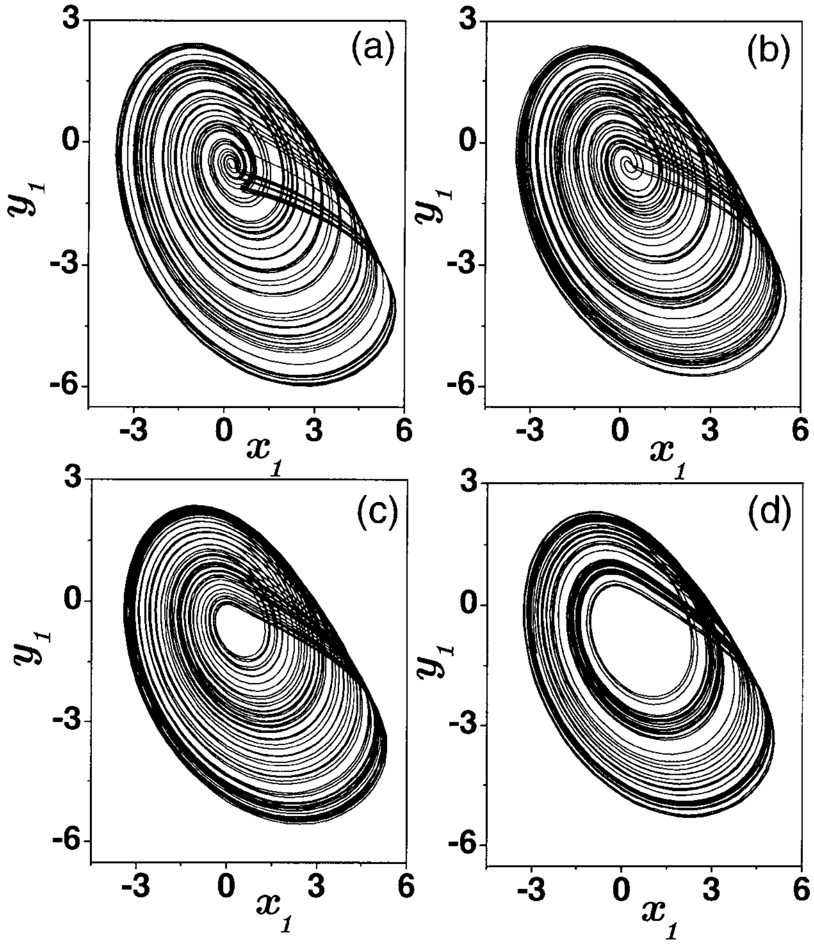

Fig. 2. Orbits of the first oscillator in the coupled system in (x, y) space. (a) $\varepsilon = 0.0$; (b) $\varepsilon = 0.008$; (c) $\varepsilon = 0.014$; (d) $\varepsilon = 0.020$.

the most interesting. In this paper, we shall look into this key transition, and try to make clear what happens there.

We consider the change of one of the chaotic attractors as the coupling strength increases, and plot the attractors for $\varepsilon = 0$, 0.008, 0.014, and 0.020 in Figures 2a–d. It is obvious that the topological structure of the attractors [10] does change when the coupling is increased above ε_0. The attractors lose their finite structure around the spiral center, where a funnel attractor (see Figs. 2a and b) changes into a simpler spiral one (see Figs. 2c and d). This topological transition is induced only by the coupling interaction.

A coupling-induced topological transition occurs, which brings forward a natural question: will the strange attractors, which are equal for very small coupling, change and become different when the coupling increases to $\varepsilon > \varepsilon_0$? Yuan *et al.* [11] proposed an idea to test if two chaotic one dimensional processional processes are dynamically identical. A single Rössler system is the same as the system considered in their Letter, and it can also be thought of as approaching asymptotically to a two dimensional chaotic attractor. Enlightened by their idea, we take a simpler way to decide when two strange chaotic attractors are not equal. Using $x_i = 0$ as a surface of section, we record the y coordinate every time (x_i, y_i, z_i) intersects $x_i = 0$ with $\mathrm{d}x_i/\mathrm{d}t > 0$. Thus we obtain a sequence $\{y_i(N)\}$, and for each case $i = 1, 2, ..., n$, we have N of them. We adopt the natural measure μ (*i.e.*, the measure of an open interval C_i is the fraction of iterates that a typical trajectory spends in C_i) [12], while $\mu(y_i)$ is the fraction of iterates the system spends anywhere (among all the data in the sequence $\{y_i(n)\}$) with $y_i(n)$ being smaller than y_i. In this way, we plot $\mu(y_i)$ *versus* y_i, $i = 1, 2, ..., n$; they should overlap with each other if the strange attractors are equal. The results for $\varepsilon = 0.008, 0.020$ are presented in Figures 3a and b, respectively. It is clearly shown in Figure 3b that the n oscillators have different natural measure, therefore they have different topological structure.

To clarify further, we define the inhomogeneity quantity as

$$Q_I = \max_{i>j}\{\max_{y_{i,j}\in[0,1]} |\mu(y_i) - \mu(y_j)|\} \qquad i, j = 1, 2, ..., n \tag{2}$$

which measures the degree of the inequality of the various attractors (also, translational symmetry). If all the attractors are equal (translational invariant) $Q_I = 0$; otherwise, $Q_I > 0$. Figure 3c shows the inhomogeneity quantity Q_I as

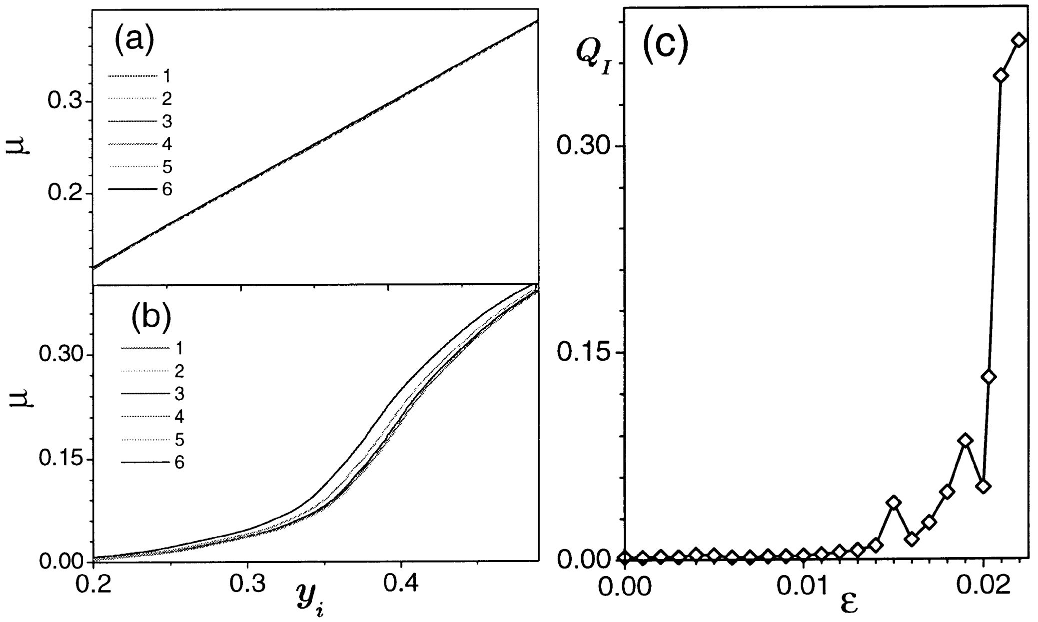

Fig. 3. Probability distributions of the natural measure for all of the n oscillators. (a) $\varepsilon = 0.008$. (b) $\varepsilon = 0.020$. Both the horizontal and the vertical axis are rescaled into $[0, 1]$, and we plot only half data for clear comparison. (c) "Inhomogeneity quantity" Q_I *versus* the coupling strength. Translational symmetry is satisfied when $\varepsilon < \varepsilon_0$, and is broken at $\varepsilon = \varepsilon_0$.

a function of the coupling strength. For $\varepsilon < \varepsilon_0$, $Q_I = 0$; for $\varepsilon > \varepsilon_0$, $Q_I > 0$, the oscillators become unequal, which reflects that the translational symmetry is broken up. Thus, at $\varepsilon \approx \varepsilon_0$, the topological transition results in the oscillators being different from one another, *i.e.*, translational symmetry breaking.

Apart from the difference in the measure of the attractors we expect that the correlation functions, for any one of the oscillators with coupling larger than ε_0, should also be different from that for $\varepsilon < \varepsilon_0$, since there, a coupling induced topological transition occurs. We define the standard correlation function for the first oscillator as

$$S_{11(12)}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x_1(t)x_{1(2)}(t-\tau)\mathrm{d}t - \lim_{T\to\infty} \frac{1}{T} \int_0^T x_1(t)\mathrm{d}t \lim_{T\to\infty} \frac{1}{T} \int_0^T x_{1(2)}(t-\tau)\mathrm{d}t \quad (3)$$

where $S_{11}(\tau)$ describes the auto-correlation function of the first oscillator with time lag τ, and $S_{12}(\tau)$ describes the correlation function between the first oscillator and the second one with time lag τ. We calculate $S_{11}(\tau)$ and $S_{12}(\tau)$ for the states shown in Figure 2, and present the results in Figures 4a1–a4 and b1–b4. The correlation functions are not always in a smooth exponentially decay profile. However, we can consider the contour lines by fitting them into exponentially decay profiles, since roughly, the smaller the time lag the larger the correlation strength, see Figures 4a1–b4. In order to give a rough measure of the characteristic decay time, we consider the quantity τ_0, which satisfies

$$S_{11(12)}(\tau_0) \approx [\max_{\tau\in(0,\infty)} S_{11(12)}(\tau)]e^{-1}. \quad (4)$$

The auto-correlation without time lag is always the maximum among all of those with $\tau \neq 0$ and τ_0 is obtained on the base of exponentially decay fitting. The characteristic time τ_0 of the auto-correlation is plotted as a function of the coupling strength ε in Figure 4c. It is reasonable that the characteristic time τ_0 for a funnel attractor is different from that of a spiral one. Increasing the coupling strength starting from zero, the characteristic time τ_0 increases slowly for small ε ($\varepsilon < \varepsilon_0$), while it does so dramatically after ε_0. It approaches to infinity as further increasing the coupling from the motion in Figure 1b, which shows that there has established approximately infinitely long range temporal correlation.

It is interesting to consider how the correlation function between one oscillator and its nearest neighbor behaves with respect to the auto-correlation. With very small coupling interaction, there is the same very small correlation between the oscillators. Increasing the coupling strength, both the strength and the characteristic time increase as shown in Figures 4b1–b4. When $\varepsilon = 0.020$, the strength and the characteristic time of $S_{12}(\tau)$ are comparable with those of $S_{11}(\tau)$. The chaotic averaged-antiphase generator happens when $\varepsilon > \varepsilon_0$, roughly at $\frac{2\pi}{n}$ angle phase lag between one oscillator and its nearest neighbor. In Figure 4d, we plot $\max(S_{12}(\tau))$ and $S_{12}(\frac{2\pi}{n})$ ($S_{12}(\frac{2\pi}{n})$ indicates the larger one between $S_{12}(\frac{2\pi}{n})$ and $S_{12}(-\frac{2\pi}{n})$ in view of the two different orientational type of the roughly spatial order). With very small coupling, almost no correlation can be detected; around ε_0, $\max(S_{12}(\tau))$ is obviously larger than $S_{12}(\frac{2\pi}{n})$, the largest correlation is not taking place at the roughly spatial order we are concerned with; with $\varepsilon > \varepsilon_0$, $S_{12}(\frac{2\pi}{n})$ increases to approach $\max(S_{12}(\tau))$ continuously, which means that the rough spatial order of

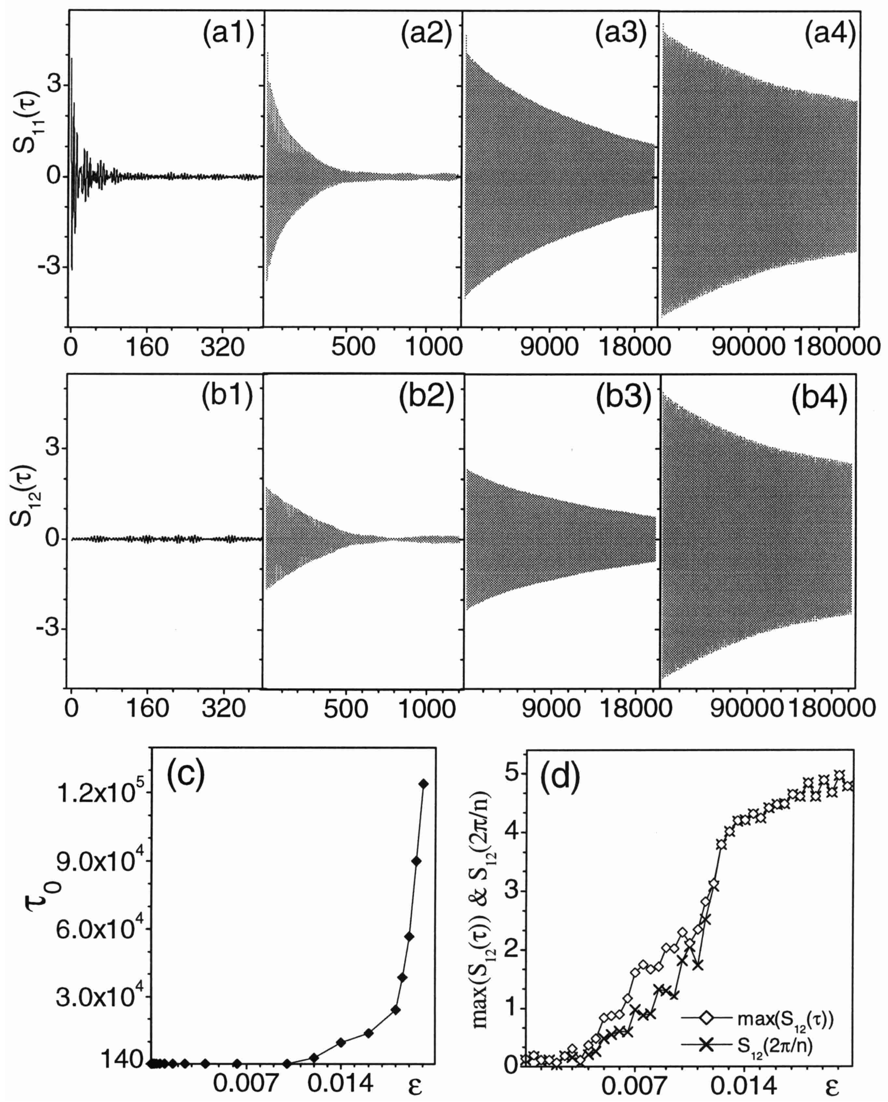

Fig. 4. (a1–a4) Auto-correlation functions $S_{11}(\tau)$ *versus* τ corresponding to Figure 2a–d. (a1) $\varepsilon = 0.0$; (a2) $\varepsilon = 0.008$; (a3) $\varepsilon = 0.014$; (a4) $\varepsilon = 0.020$. (b1–b4) Correlation functions $S_{12}(\tau)$ *versus* τ corresponding to (a1–a4). (c) The characteristic time τ_0 defined in equation (3) for auto-correlation function *versus* the coupling strength ε. (d) $\max(S_{12}(\tau))$ and $S_{12}(\frac{2\pi}{N})$ *versus* ε. $S_{12}(\frac{2\pi}{N})$ corresponds to the larger one between $S_{12}(\frac{2\pi}{N})$ and $S_{12}(-\frac{2\pi}{N})$.

averaged antiphase is that with the largest correlation between nearest oscillators since $S_{12}(\frac{2\pi}{n}) = \max(S_{12}(\tau))$, in this way strengthening the argument of comparing measures. Having had $S_{12}(\frac{2\pi}{n}) = \max(S_{12}(\tau))$, the roughly spatial order has been established as the antiphase state.

In summary, we investigate the high-dimensional chaotic model of diffusively coupled ring of Rössler oscillators. An averaged-antiphase state generates from the high-dimensional chaotic sea around ε_0, and will develop into a periodic state of conventional rotating wave by increasing the coupling continuously. This transition around ε_0 is our focus in the present work. With $\varepsilon < \varepsilon_0$, the oscillators have a common strange attractor and they evolve "randomly" in both time and spatial. Around ε_0, there happens a topological transition induced by the coupling. The topological transition not only changes the topological structure of all the oscillators, which share one common strange attractor, but also changes them into being different from each other. On the basis of auto-correlation functions, we show that starting from this transition, the system begins to establish infinitely long range temporal correlation. On the other hand, the correlation function between neighboring oscillators show that the angular organization is around $\frac{2\pi}{n}$ phase lag, which is the very generator of a rotating wave.

To spot the transitions, we propose a simple way to determine if two strange chaotic attractors are not equal, thus showing for the first time that the elements of a coupling system have different topological structure. As reviewed in the introduction, the periodic rotating wave can be expected in the three kinds of coupling chaotic systems which possess Z_n, D_n, or S_n symmetry. We hope the topological transition demonstrated in this paper can be found useful in finding rotating waves in experimental systems of the three common types of chaotic rings.

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