Transition to complete synchronization in phase-coupled oscillators with nearest neighbor coupling

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We investigate synchronization in a Kuramoto-like model with nearest neighbor coupling. Upon analyzing the behavior of individual oscillators at the onset of complete synchronization, we show that the time interval between bursts in the time dependence of the frequencies of the oscillators exhibits universal scaling and blows up at the critical coupling strength. We also bring out a key mechanism that leads to phase locking. Finally, we deduce forms for the phases and frequencies at the onset of complete synchronization. © *2009 American Institute of Physics.*
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Weakly coupled oscillators play an important role in understanding collective behavior of large populations. They are often used to model the dynamics of a variety of systems that arise in nature, even though they are quite different. Synchronization is one of the interesting phenomena observed in these systems where the interacting oscillators under the influence of coupling would have a common frequency. Particularly, these systems show an extremely complex clustering behavior as a function of the coupling strength. In spite of their differences, the above-mentioned systems can be described using simple models of coupled phase equations such as the Kuramoto model. This paper analyzes the behavior of individual oscillators in the vicinity of the critical coupling where all the oscillators evolve in synchrony with each other.

I. INTRODUCTION

Systems of coupled oscillators can describe problems in physics, chemistry, biology, neuroscience, and other disciplines. They have been widely used to model several phenomena, such as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, and neurodynamics.[1–3] These systems have been observed to synchronize themselves to a common frequency when the coupling strength between these oscillators is increased.[4–13] The synchronization features of many of the above-mentioned systems, in spite of the diversity of the dynamics, might be described using simple models of weakly coupled phase oscillators such as the Kuramoto model.[8,14]

Finite range interactions are more realistic for the description of many physical systems, although finite range coupled systems are difficult to analyze and to solve analytically. However, in order to figure out the collective phenomena when finite range interactions are considered, it is of fundamental importance to study and to understand the nearest neighbor interactions, which is the simplest form of the local interactions. In this context, a simplified version of the Kuramoto model with nearest neighbor coupling in a ring topology, which we shall refer to as the locally coupled Kuramoto model (LCKM), is a good candidate to describe the dynamics of coupled systems with local interactions such as Josephson junctions, coupled lasers, neurons, chains with disorders, multicellular systems in biology, and in communication systems.[6,14–16] For instance, it has been shown that the equations of the resistively shunted junction which describe a ladder array of overdamped, critical-current disordered Josephson junctions that are current biased along the rungs of the ladder can be expressed by a LCKM.[17] In nearest neighbor coupled Rössler oscillators, the phase synchronization can be described by the LCKM.[18] Therefore, LCKM can provide a way to understand phase synchronization in coupled systems, for example, in locally coupled lasers,[19,20] where local interactions are dominant. Coupled phase oscillators described by LCKM can also be used to model the occurrence of travelling waves in neurons.[6,21] In communication systems, unidirectionally coupled Kuramoto model can be used to describe an antenna array.[22] Such unidirectionally coupled Kuramoto models can be considered as a special case of the LCKM and it often mimics the same behavior.

One of the important features of the local model is that the properties of individual oscillators can be easily analyzed to study the collective dynamics while one has to rely on the average quantities, in a mean field approximation, or by means of an order parameter, etc., as in the case of the usual

Kuramoto model of long range interactions. Therefore, due to the difficulty in applying standard techniques of statistical mechanics, one should look for a simple approach to understand the coupled system with local interactions by means of numerical study of a temporal behavior of the individual oscillators. Such analysis is necessary in order to obtain a close picture of the effect of the local interactions on synchronization. In this case, numerical investigations can assist one to figure out the mechanism of interactions at the stage of complete synchronization, which in turns help to obtain an analytic solution. Earlier studies on the LCKM show several interesting features including tree structures with synchronized clusters, phase slips, bursting behavior, saddle-node bifurcation, and so on.[23–25] There have been studies showing that neighboring elements share dominating frequencies in their time spectra, and that this feature plays an important role in the dynamics of formation of clusters in the local model.[26,27] It has been found that the order parameter, which measures the evolution of the phases of the nearest neighbor oscillators, becomes maximum at the partial synchronization points inside the tree of synchronization.[28] Very recently, we developed a scheme based on the method of Lagrange multipliers to estimate the critical coupling strength for complete synchronization in the local Kuramoto model with different boundary conditions.[29]

In this paper we address the mechanism that leads to a complete synchronization in the Kuramoto model with local coupling. This is done by analyzing the behavior of each individual oscillators at the onset of synchronization. For this purpose we consider the equations governing the phase differences at the onset of synchronization. In particular, we identify that the cosine of only one among the phase differences becomes zero. Based on this property we derive the expression for the time interval between bursting behavior of the instantaneous frequencies of each individual oscillators in the vicinity of critical coupling strength. Our analysis shows that the transition to complete synchronization occurs due to a saddle-node bifurcation in agreement with the earlier studies. Further, we deduce the expressions for the phases and frequencies of the individual oscillators at the onset of complete synchronization.

This paper is organized as follows. In Sec. II we present a brief overview on the dynamics of local Kuramoto model. We then analyze the behavior of the phase differences and the time interval between successive bursts at the transition to complete synchronization. In particular, we point out the mechanism that lead to complete phase locking at the critical coupling strength. Based on this, we deduce the forms of phases and frequencies at the onset of synchronization. Finally, in Sec. III we give a summary of the results and conclusions.

II. BEHAVIOR OF PHASES AND FREQUENCIES AT THE ONSET OF SYNCHRONIZATION

Even when there has been an extensive exploration of the dynamics of the Kuramoto model (global coupling among all oscillators), the local model of nearest neighbor interactions, which can be considered as a diffusive version
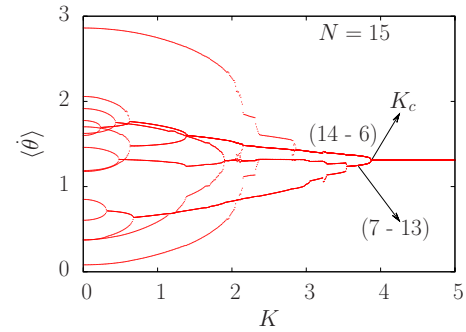

FIG. 1. (Color online) Synchronization tree for a system of 15 oscillators.

of the Kuramoto model, has been receiving attention only recently. The LCKM is expressed as[25–29]

$$\dot{\theta}_i = \omega_i + \frac{K}{3}[\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)], \quad (1)$$

where ω_i are the natural frequencies, K is the coupling strength, θ_i is the instantaneous phase, $\dot{\theta}_i$ is the instantaneous frequency, and $i = 1, 2, \ldots, N$. Many interesting features of the LCKM remain unknown, especially an analytic solution,[14] which would be of great importance in understanding the mechanism that leads to synchronization. In order to find such an analytic solution, one should study carefully the temporal evolution of frequency and phase of each individual oscillator in the neighborhood of the critical coupling for complete synchronization.

If we consider the oscillators in a ring, with periodic boundary conditions $\theta_{i+N} = \theta_i$, the nonidentical oscillators (1) cluster in time averaged frequency until they completely synchronize to a common value of the average frequency $\omega_0 = (1/N)\Sigma_{i=1}^{N}\omega_i$, at a critical coupling K_c.[24–26,28,29] At $K \geq K_c$, the phases and the frequencies are time independent and all the oscillators remain synchronized. In Fig. 1, we show the synchronization tree for a periodic system with $N = 15$ oscillators, where the elements which compose each one of the major clusters that merge into one at K_c are indicated in each branch.

In terms of phase differences $\phi_i = \theta_{i+1} - \theta_i$, system (1) can be rewritten as

$$\dot{\phi}_i = \omega_{i+1} - \omega_i + \frac{K}{3}[\sin \phi_{i-1} - 2 \sin \phi_i + \sin \phi_{i+1}], \quad (2)$$

with $\dot{\phi}_i^* = 0$ at K_c for $i = 1, 2, \ldots, N$. In addition, all quantities $\dot{\phi}_i^*$, ϕ_i^*, and $\dot{\theta}_i^*$, which become time independent[25,26,28,29] at the critical coupling, remain like that for $k \geq K_c$, when $\dot{\phi}_i^* = 0$ and $\dot{\theta}_i^* = \omega_0$. Earlier attempts to obtain a solution of Eq. (2) show that for only two oscillators which have phase difference $\phi_l^* = \theta_l^* - \theta_{l-1}^*$, results in $|\sin \phi_l^*| = 1$,[23] and indeed this is a necessary condition for Eq. (2) to have a phase-locked solution. This fact has been used by Daniels *et al.*[17] to estimate the value of critical coupling strength K_c at which the transition to complete synchronization occurs. However, the determination of which two oscillators among N oscillators that have $|\sin \phi_l^*| = 1$, remains difficult. From the study of the temporal evolution of phases and frequencies of each indi-
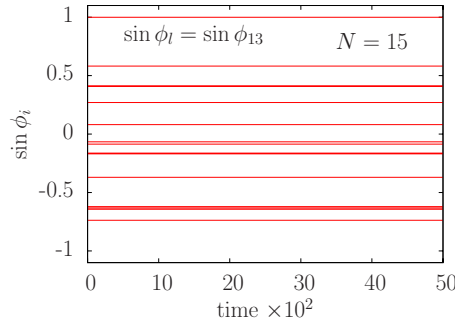
FIG. 2. (Color online) Values of $\sin \phi_i$ at $K \gtrsim K_c$ for a system of 15 oscillators (see Fig. 1).

TABLE I. Calculated values of K_c and A for different values of N.

N	K_c	A
3	0.850 412 27	1.9994
5	3.170 827 13	2.0001
10	3.547 010 35	1.9996
15	3.870 238 66	2.0000
20	4.958 300 14	2.0002
25	3.641 060 38	1.9989
50	9.457 200 49	1.9993
100	12.723 208 7	1.9985

vidual oscillator, it has also been found numerically that, at the onset of synchronization $K \lesssim K_c$, the values of $\dot{\theta}_i(t)$ and $\dot{\phi}_i(t)$ remain equal to ω_0 and zero, respectively, for a certain time interval T. During this time T, a stable phase-locked solution exists; then they burst,[18,24,25] and this stable phase-locked solution is lost. In between bursts, the phases remain fixed and then they have an abrupt change (phase-slip behavior) by an amount which depends on the initial values of the frequencies ω_i,[24,25] corresponding to the burst in the frequencies, while the quantities $\Sigma_{i=1}^{N} \phi_i = 0$ and $\Sigma_{i=1}^{N} \dot{\phi}_i = 0$ are always preserved by the topology. Integrated with the above information, it has been shown by numerical investigation that the time interval T blows up as K becomes close to K_c and $T \to \infty$ at K_c. All this information leads one to conclude that there is a saddle-node bifurcation at K_c, and the synchronization-desynchronization transition at the critical coupling can be interpreted using this knowledge.

In this work, we perform numerical investigations of the temporal evolution of the phases and frequencies for the individual oscillators in order to arrive to specific conditions which will lead to criteria to obtain an analytic solution. A detailed study of all quantities $\sin \phi_i^*$ at K_c for several values of N and for different sets of ω_i, shows that there is only one value of phase difference between two neighboring oscillators $\phi_l^* = \theta_{l+1}^* - \theta_l^*$ for which $|\sin \phi_l^*| = 1$, while for all other values $|\sin \phi_i^*| \neq 1$, $i \neq l$. In Fig. 2, we show $\sin \phi_i^*$ for a case of $N = 15$ as time progresses at the critical coupling K_c, with the same initial frequencies of Fig. 1. We see that the value of $|\sin \phi_l^*| = 1$, is for $l = 13$ and that this quantity $|\sin \phi_l^*| = 1$ holds for only one value of phase difference $\phi_l = \pi/2$, where these two oscillators $l+1$ and l belong to different clusters, and these two nearest neighbors oscillators are always at the borders between the major clusters that merge at K_c, which can be seen from Fig. 1. We find the same result for different initial frequencies ω_i and for different values of N. In addition, the sign of $\sin \phi_l^*$ is negative for $\omega_l > \omega_{l+1}$ and positive for the reverse.

The knowledge of the burst and phase slip (in the vicinity of K_c) of the quantities $\dot{\phi}_i(t)$ and $\phi_i(t)$, respectively, as well as the finding of $|\sin \phi_l^*| = 1$ (at K_c), will allow us to rewrite equation (2), for the index l as

$$\dot{\phi}_l = B(A - 2 \sin \phi_l), \tag{3}$$

where $A = 3|(\omega_{l+1} - \omega_l)|/K + \sin \phi_{l-1} + \sin \phi_{l+1}$ and $B = K/3$. Equation (3) takes the form of a phase synchronization of two coupled limit-cycles.[30] At K_c, $\phi_l^* = \pm \pi/2$, ϕ_{l-1}^* and ϕ_{l+1}^* are constants and time independent, and $A = 2$. A detailed numerical study shows also that, at the onset of synchronization, $A \simeq 2$ and the values of ϕ_l, ϕ_{l-1}, and ϕ_{l+1} remain equal to their values at K_c, for a time interval T. The values of K_c and A for different numbers of oscillators N from numerical simulations are tabulated in Table I. It is clear that in all the cases, $A \simeq 2$ when K approaches K_c. The relation $A \simeq 2$ is found to be valid for different choices of initial frequencies ω_i for each N in the vicinity of K_c. Further, it should be noted that when the time interval $T \to \infty$, one can find that $A = 2$. The time interval T can be found analytically, according to Eq. (3), to be

$$T \approx \frac{3\pi\sqrt{2}}{K\sqrt{A}} \frac{1}{\sqrt{A-2}}. \tag{4}$$

In Fig. 3, we clearly see that T blows up as A becomes close to 2 (where K goes to K_c), for the case of $N = 15$. We find that T blows up as $(A-2)^{-0.5}$, which is a numerical proof that a saddle-node bifurcation occurs at K_c. Assuming that $\sin \phi_{l-1}$ and $\sin \phi_{l+1}$ remain constant for a time interval T in the vicinity of K_c, and are equal to their values at K_c (which has been verified numerically), we find that $AK/2 \approx K_c$. Table II shows this fact where the error is small and decreases as K approaches K_c. Therefore, Eq. (4) takes the form
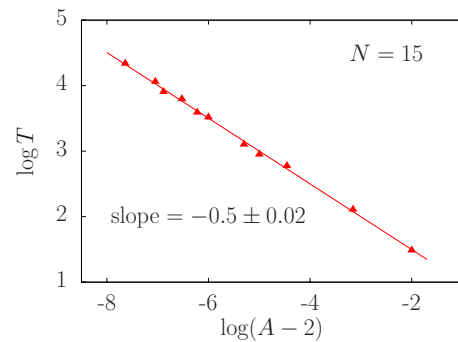

FIG. 3. (Color online) $\log_{10} T$ vs $\log_{10}(A-2)$, which shows the divergence of the time interval T when A approaches 2 with a slope $\simeq -0.5$.

TABLE II. Calculated values of $AK/2$ for $N=15$ oscillators at the vicinity of $K_c=3.870\,238\,658$.

K	A	$\dfrac{AK}{2}$	$|K_c-K|$
3.869 480 136	2.000 700 0	3.870 834 454	5.960×10^{-4}
3.870 198 414	2.000 035 0	3.870 266 142	2.750×10^{-5}
3.870 226 709	2.000 010 0	3.870 246 060	7.402×10^{-6}
3.870 237 122	2.000 001 0	3.870 238 909	2.480×10^{-7}
3.870 238 325	2.000 000 3	3.870 238 727	6.920×10^{-8}

$$T \approx \frac{3\pi}{\sqrt{2}\sqrt{K_c}\sqrt{K_c-K}}. \tag{5}$$

Thus, within a good approximation, the periodic time interval T blows up as $(K_c-K)^{-0.5}$, in good agreement with the numerical calculation by Zheng *et al.*,[24,25] showing that a saddle-node bifurcation occurs at K_c.[23]

Therefore, Eq. (3) can be written as

$$\dot{\phi}_l \approx L(K_c - K\sin\phi_l), \tag{6}$$

which can be solved analytically, and its solution reads

$$\phi_l \approx 2\arctan\left[\frac{\alpha\tan\left(\frac{1}{2}\alpha Lt\right)\pm K}{K_c}\right] \tag{7a}$$

and

$$\dot{\phi}_l \approx \frac{L}{K_c}\frac{\alpha^2\sec^2\left(\frac{1}{2}\alpha Lt\right)}{1+\left\{\frac{1}{K_c}\left[\alpha\tan\left(\frac{1}{2}\alpha Lt\right)\pm K\right]\right\}^2}, \tag{7b}$$

where $\alpha=\sqrt{K_c^2-K^2}$ and $L=\frac{2}{3}$. Equations (7a) and (7b) show that, at K_c, the values $\sin\phi_l^*=\pm1$, which lead to $\dot{\phi}_l^*=0$. It can also be seen that in the vicinity of K_c, $\sin\phi_l=\pm1$ and $\dot{\phi}_l=0$ for a period T. The $(+)$ sign in Eqs. (7a) and (7b) corresponds to the case $\omega_{l+1}>\omega_l$ and the $(-)$ sign for the reverse.

In order to understand the mechanism of full synchronization which occurs at K_c, we use the fact that $\sin\phi_l^*=\pm1$ and each $\dot{\theta}_i^*=\omega_0$, where these quantities remain unchanged for T in the vicinity of K_c. Hence, from system (1), we are able to obtain the following relations:

$$\sin\phi_{l+m}^* = \frac{3}{K_c}\sum_{m=1}^{N-l}(\omega_0-\omega_{l+m})\pm\sin\phi_l^*, \tag{8a}$$

$$\sin\phi_{l-n}^* = -\frac{3}{K_c}\sum_{n=1}^{l-1}(\omega_0-\omega_{l-n-1})\pm\sin\phi_l^*. \tag{8b}$$

Using this fact, we write the following equations, in addition to Eqs. (7a) and (7b):

$$\phi_{l-n} \approx \sin^{-1}(a_n\pm\sin\phi_l), \tag{9a}$$

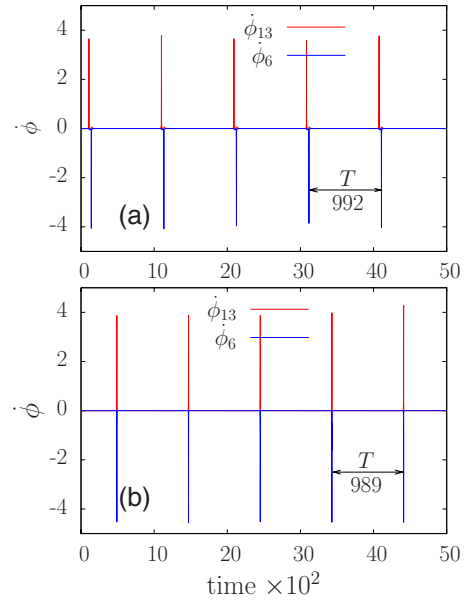$$\dot{\phi}_{l-n} \approx \frac{\cos\phi_l\dot{\phi}_l}{\sqrt{1-(a_n\pm\sin\phi_l)^2}}, \tag{9b}$$

FIG. 4. (Color online) Time evolution of $\dot{\phi}_{13}$ and $\dot{\phi}_6$ according to (a) system (1) and (b) Eqs. (7b) and (9b), at $K=3.870\,226\,709$, for 15 oscillators with the same initial conditions of Fig. 1.

$$\phi_{l+m} \approx \sin^{-1}(a_m\pm\sin\phi_l), \tag{9c}$$

$$\dot{\phi}_{l+m} \approx \frac{\cos\phi_l\dot{\phi}_l}{\sqrt{1-(a_m\pm\sin\phi_l)^2}}, \tag{9d}$$

where $a_n=(-3/K_c)\Sigma_{i=1}^n(\omega_0-\omega_{l-i-1})$ with $n=1,2,3,\ldots,l-1$ and $a_m=(3/K_c)\Sigma_{j=1}^m(\omega_0-\omega_{l+j})$ with $m=1,2,3,\ldots,N-l$. It is clearly seen that according to the above equation, each ϕ_i can be expressed in terms of ϕ_l and, consequently, each $\dot{\phi}_i$ can be expressed in terms of ϕ_l and $\dot{\phi}_l$. Therefore, all values of ϕ_i will be shifted from each other by some constant which is determined by the location of the indexes $l-n$ and $l+m$ relative to oscillators with indexes l and $l+1$. This is shown in Fig. 2, where $\sin\phi_i$ values are shifted from each other at K_c. Therefore, at K_c, what occurs to ϕ_l and $\dot{\phi}_l$ due to saddle-node bifurcation diffuses through the ring via interaction between neighboring oscillators. This means that, at the vicinity of K_c, the value of ϕ_l has an abrupt change after being constant for a time T, caused by a burst behavior of $\dot{\phi}_l$ after being zero for the same time interval T. The abrupt change in ϕ_l produces a sudden change in the values of ϕ_i of their neighbors, while the bursting behavior of $\dot{\phi}_l$ in turn yields bursts in $\dot{\phi}_i$ $(i\neq l)$. In order to demonstrate this fact, we plot the temporal evolution of $\dot{\phi}_{13}$ and $\dot{\phi}_6$, in the vicinity of K_c, according to numerical simulation of Eq. (1) in Fig. 4(a), while we plot both quantities according to Eqs. (7b) and (9b) in Fig. 4(b). As shown in Fig. 4, the results of numerical simulation agree with that from the analytic solution. The above-mentioned behavior is reflected in the time dependence of the $\dot{\theta}_i's$, which in turn remain equal to ω_0 for a time T and burst around ω_0 corresponding to the burst of ϕ_l. Henceforward, we argue that it is the behavior of ϕ_l and $\dot{\phi}_l$ which drives the system to fall into full synchronization.

III. SUMMARY AND CONCLUSIONS

In summary, we have analyzed the conditions on the phase differences for the onset of complete synchronization at the critical coupling strength in a Kuramoto-like model with nearest neighbor coupling. Such a condition, which is $|\sin \phi_l^*|=1$ (or $\cos \phi_l^*=0$), allows us to solve the equations of the phase differences [Eq. (2)] analytically. We also found that full synchronization occurs always when the quantity $A=2$ at K_c. Due to the diffusive nature of the LCKM, complete synchronization of all oscillators to a common value can be interpreted and understood once we have analytic forms for ϕ_l and $\dot{\phi}_l$. However, it is still difficult to determine analytically the number of oscillators in each cluster which merge into one at K_c. Therefore, one cannot allocate straightforwardly the two nearest neighbor oscillators which would have $|\sin \phi_l^*|=1$. On the other hand, a detailed numerical study on the temporal evolution of phases, phase differences, and frequencies of oscillators at the borders of the clusters that merge into larger one at onset of complete synchronization helps us to determine the neighboring oscillators which have $\sin \phi_l^* = \pm 1$. Such analysis can also be used to understand the partial synchronization that leads to the formation of small clusters for coupling strengths below the critical coupling strength K_c. Of course, analysis of the simplest case of locally coupled phase oscillators can help to understand models with local interactions where amplitudes and phases are included.[15,18–20] In such cases, a detailed study of the time evolution of amplitudes and phases can reveal a better understanding of the mechanism of synchronization. The present analysis can also be applicable to models in higher dimensions such as that for dislocations in solids which includes local nearest neighbor interactions.[31] Furthermore, the present approach can be extended to understand the underlying mechanism in the case of locally coupled Kuramoto models with time delay[6] (or phase delay) introduced between the coupled oscillators. In addition, the mechanism of synchronization in LCKM for open and fixed boundaries can be studied in a similar manner to the present work as well as for the case of unidirectional LCKM. We also want to mention that the scaling law given by Eq. (5) has been found experimentally in a transition to phase synchronization in CO_2 lasers[32] and in electronic circuits.[33,34] On the other hand, one cannot make a direct comparison between the mechanism of synchronization discussed here in LCKM and the scaling law that has been found in experiments since the physical systems are not necessarily the same.

[1]A. T. Winfree, *Geometry of Biological Time* (Springer, New York, 1990).
[2]C. W. Wu, *Synchronization in Coupled Chaotic Circuits and Systems* (World Scientific, Singapore, 2002).
[3]S. H. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2003).
[4]C. M. Gray, P. Koenig, A. K. Engel, and W. Singer, Nature (London) **338**, 334 (1989).
[5]K. Otsuka, *Nonlinear Dynamics in Optical Complex Systems* (Kluwer, Dordrecht, 2000).
[6]H. Haken, *Brain Dynamics: Synchronization and Activity Patterns in Pulse-Coupled Neural Nets with Delays and Noise* (Springer, Berlin, 2007).
[7]*Bifurcation, Patterns, and Symmetry*, Special Issue of Physica D, edited by M. Golubitsky and E. Knobloch, Vol. **143** (2000).
[8]Y. Kuramoto, *Chemical Oscillations, Waves and Turbulences* (Springer, Berlin, 1984).
[9]G. Hu, Y. Zhang, H. A. Cerdeira, and S. Chen, Phys. Rev. Lett. **85**, 3377 (2000).
[10]Y. Zhang, G. Hu, H. A. Cerdeira, S. Chen, T. Braun, and Y. Yao, Phys. Rev. E **63**, 026211 (2001).
[11]Y. Zhang, G. Hu, and H. A. Cerdeira, Phys. Rev. E **64**, 037203 (2001).
[12]I. A. Heisler, T. Braun, Y. Zhang, G. Hu, and H. A. Cerdeira, Chaos **13**, 185 (2003).
[13]P. A. Tass, *Phase Resetting in Medicine and Biology* (Springer, Berlin, 1999).
[14]J. A. Acebron, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, Rev. Mod. Phys. **77**, 137 (2005).
[15]Y. Ma and K. Yoshikawa, arXiv:0809.1697v2, 2008.
[16]Y. Braiman, T. A. Kennedy, K. Wiesenfeld, and A. Khinik, Phys. Rev. A **52**, 1500 (1995).
[17]B. C. Daniels, S. T. M. Dissanayake, and B. R. Trees, Phys. Rev. E **67**, 026216 (2003).
[18]Z. Liu, Y.-C. Lai, and F. C. Hoppensteadt, Phys. Rev. E **63**, 055201(R) (2001).
[19]A. Khinik, Y. Braiman, V. Protopopescu, T. A. Kennedy, and K. Wiesenfeld, Phys. Rev. A **62**, 063815 (2000).
[20]D. Tsygankov and K. Wiesenfeld, Phys. Rev. E **73**, 026222 (2006).
[21]S. Manrrubia, A. Mikhailov, and D. Zanette, *Emergence of dynamical Order: Synchronization Phenomena in Complex Systems* (World Scientific, Singapore, 2004).
[22]J. Rogge and D. Aeyels, J. Phys. A **37**, 11135 (2004).
[23]S. H. Strogatz and R. E. Mirollo, Physica D **31**, 143 (1988).
[24]Z. Zheng, G. Hu, and B. Hu, Phys. Rev. Lett. **81**, 81 (1998).
[25]Z. Zheng, B. Hu, and G. Hu, Phys. Rev. E **62**, 402 (2000).
[26]H. F. El-Nashar, A. S. Elgazzar, and H. A. Cerdeira, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 2945 (2002).
[27]H. F. El-Nashar, Y. Zhang, H. A. Cerdeira, and F. Ibyinka A., Chaos **13**, 1216 (2003).
[28]H. F. El-Nashar, Int. J. Bifurcation Chaos Appl. Sci. Eng. **13**, 3473 (2003).
[29]P. Muruganandam, F. F. Ferreira, H. F. El-Nashar, and H. A. Cerdeira, Pramana, J. Phys. **70**, 1143 (2008).
[30]S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Perseus, New York, 2000).
[31]A. Carpio and L. L. Bonilla, Phys. Rev. Lett. **90**, 135502 (2003).
[32]S. Bocaletti, E. Allaria, R. Meucci, and F. Arecchi, Phys. Rev. Lett. **89**, 194101 (2002).
[33]L. Zhu, A. Raghu, and Y.-C. Lai, Phys. Rev. Lett. **86**, 4017 (2001).
[34]G.-M. Kim, G.-S. Yim, J.-W. Ryu, Y.-J. Park, and D.-U. Hwang, Europhys. Lett. **71**, 723 (2005).