Determination of the critical coupling for oscillators in a ring

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We study a model of coupled oscillators with bidirectional first nearest neighbors coupling with periodic boundary conditions. We show that a stable phase-locked solution is decided by the oscillators at the borders between the major clusters, which merge to form a larger one of all oscillators at the stage of complete synchronization. We are able to locate these four oscillators depending only on the set of the initial frequencies. Using these results plus an educated guess (supported by numerical findings) of the functional dependence of the corrections due to periodic boundary conditions, we are able to obtain a formula for the critical coupling, at which the complete synchronization state occurs. Such formula fits well in very good accuracy with the results that come from numerical simulations. This also helps to determine the sizes of the major clusters in the vicinity of the stage of full synchronization. © *2009 American Institute of Physics*.
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Weakly coupled oscillators in the chaotic state have been known to represent many physical systems as well as chemical, biological, neurological, and so on. These systems synchronize in frequency under the influence of coupling. Knowing beforehand the value of the coupling constant and the dynamical behavior of the individual oscillators for complete synchronization to occur is an important source of information for real applications. This paper is a continuation of previous theoretical results for these systems. Here, we derive relationships that allow us to determine the oscillators which first lock in phase and drag the whole system into the synchronized state as well as the size of the two existing clusters before the transition.

I. INTRODUCTION

In recent years we have seen oscillators coupled through nearest neighbor interactions to be used to understand the behavior of systems in physics, chemistry, biology, neurology, as well as other disciplines, and to model several phenomena such as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, and neurodynamics.[1–4] These systems have been observed to synchronize themselves to a common frequency when the coupling strength between these oscillators is increased.[4–6] In spite of the diversity of the dynamics, the synchronization features of many of the above mentioned systems might be described using a simple model of weakly coupled phase oscillators such as the Kuramoto model,[4–7] as well as its variations to adapt it for finite range interactions which are more realistic to mimic many physical systems. Difficulties arise since finite range coupled systems are difficult to analyze and to solve analytically. In spite of that, in order to figure out the collective phenomena when finite range interactions are considered, it is of fundamental importance to study and to understand the nearest neighbor interactions, which is the simplest form of the local interactions. In this context, a simplified version of the Kuramoto model with nearest neighbor coupling in a ring topology, which we shall refer to as locally coupled Kuramoto model (LCKM), is a good candidate to describe the dynamics of coupled systems with local interactions. Several reports exist where the LCKM has been used to represent the dynamics of a variety of systems such as Josephson junctions, coupled lasers, neurons, chains with disorders, and multicellular systems in biology and in communication systems.[5,7–9] It has also been shown that the equations of the resistively shunted junction which describe a ladder array of overdamped, critical-current disordered Josephson junctions that are current biased along the rungs of the ladder can be expressed by a LCKM.[10] For nearest neighbor coupled Rössler oscillators the phase synchronization can be described by the LCKM,[11] as well as locally coupled lasers,[12,13] where local interactions are dominant. LCKM can also be used to model the occurrence of traveling waves in neurons.[4,5] In communication systems, unidirectionally coupled Kuramoto model can be used to describe an antenna array.[14] Such unidirectionally coupled Kuramoto models can be considered as a special case of the LCKM and it often mimics the same behavior. Therefore, LCKM can provide a way to understand phase synchronization in coupled systems in general.

While in the Kuramoto model for long range interactions one has to rely on average quantities, in a mean field approximation or by means of an order parameter, etc., in the

local model it is necessary to study the behavior of individual oscillators in order to understand the collective dynamics. Therefore, due to the difficulty in applying standard techniques of statistical mechanics, one should look for a simple approach to understand the coupled system with local interactions by means of numerical study of the temporal behavior of the individual oscillators. Such analysis is necessary in order to obtain a close picture of the effect of the local interactions on synchronization. In this case, numerical investigations can assist to figure out the mechanism of interactions at the stage of complete synchronization which in turn helps to get an analytic solution. Earlier studies on the LCKM show several interesting features including tree structures with synchronized clusters, phase slips, bursting behavior and saddle node bifurcation, and so on.[15,16] It has also been shown that neighboring elements share dominating frequencies in their time spectra, and that this feature plays an important role in the dynamics of formation of clusters in the local model:[17] that the order parameter, which measures the evolution of the phases of the nearest neighbor oscillators, becomes maximum at the partial synchronization points inside the tree of synchronization[18] and a scheme has been developed based on the method of Lagrange multipliers to estimate the critical coupling strength for complete synchronization in the local Kuramoto model with different boundary conditions.[19]

Very recently, we identified two oscillators which are responsible for dragging the system into full synchronization,[20] and the difference in phase for this pair is $\pm\pi/2$. These two oscillators are among two pairs of oscillators which are formed by the four oscillators at the borders between major clusters in the vicinity of the critical coupling. Using these results here we develop a method to obtain a mathematical expression for the value of the critical coupling at which full synchronization occurs, once a set of initial conditions for the frequencies of the N oscillators is assigned. In the process of finding the solution, we come across two quantities which will permit us to identify those oscillators at the borders between major clusters mentioned before, and also determine the number of oscillators at the major clusters in the vicinity of the critical coupling. Finally, with the help of the formula for the critical coupling we can identify the pair of oscillators which has the phase-lock condition, depending only on the set of the initial frequencies.

This paper is organized as follows. In Sec. II we investigate the LCKM where periodic boundary conditions are used. We determine the critical coupling at the stage of complete synchronization as well as the number of oscillators at each cluster in the vicinity of the critical coupling. Finally, in Sec. III we give a conclusion, which is based on the summary of the results.

II. OSCILLATORS IN A RING

The local model of nearest neighbor interactions or LCKM can be considered as a diffusive version of the Kuramoto model and it is expressed as[16–20]

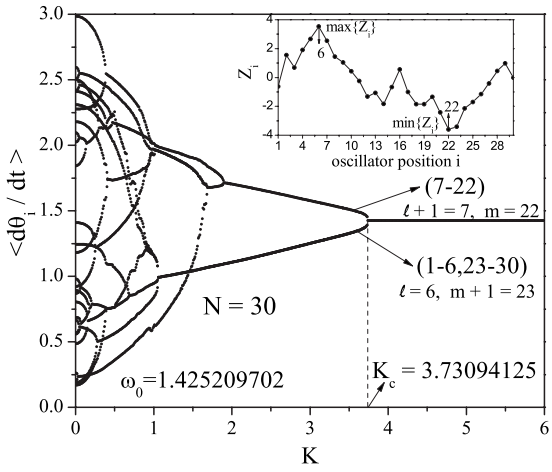

FIG. 1. Synchronization tree for a system of 30 oscillators with detailed composition of each cluster before full synchronization. The oscillators at the borders between major cluster in the vicinity of K_c are shown. The inset shows Z_i vs i, where $\max\{Z_i\}$ corresponds to Z_l and $\min\{Z_i\}$ to Z_m.

$$\dot{\theta}_i = \omega_i + K[\sin(\phi_i) - \sin(\phi_{i-1})] \tag{1}$$

with periodic boundary conditions $\theta_{i+N}=\theta_i$ and for $i=1,2,\ldots,N$. The set of the initial values of frequencies $\{\omega_i\}$ is the natural frequencies taken from a Gaussian distribution and K is the coupling strength. The phase difference is defined as $\phi_i=\theta_{i+1}-\theta_i$ for $i=1,2,3,\ldots,N$. These nonidentical oscillators (1) cluster in time averaged frequency until they completely synchronize to a common value given by the average frequency $\omega_0=(1/N)\Sigma_{i=1}^{N}\omega_i$ at a critical coupling K_c. At $K\geq K_c$ the phase differences and the frequencies are time independent and all the oscillators remain synchronized. In Fig. 1 we show the synchronization tree for a periodic system with $N=30$ oscillators, where the elements which compose each one of the major clusters are indicated in each branch. These clusters merge into one at K_c where all oscillators have the same frequency. The major clusters just before K_c contain N_1 and N_2 oscillators, where $N=N_1+N_2$. It is not necessary for these clusters to have the same numbers of oscillators. At the vicinity of K_c, major clusters of successive oscillators have sets of nearest neighbors at the borders. An interesting fact emerges: the phase-locked solution is always valid for one and only one phase difference, and this is the difference between the phases of the two oscillators at the border of the clusters.[20] Thus, for these two neighboring oscillators, the equation for the phase difference is

$$\dot{\phi}_n = \Delta_n - 2K\sin(\phi_n) + K\sin(\phi_{n-1}) + K\sin(\phi_{n+1}), \tag{2}$$

where $\Delta_n=\omega_{n+1}-\omega_n$. Equation (2) at K_c has $\dot{\phi}_n=0$, and hence $\dot{\theta}_n=\dot{\theta}_{n+1}=\omega_0$. It has been found that the phase-locked solution is satisfied when $\phi_n=\pi/2$ for the case of $\omega_{n+1}>\omega_n$ and $\phi_n=-\pi/2$ for the reverse. In addition the phase-locked solution exists when[20] $X_n=|\Delta_n/K_c+[\sin(\phi_{n+1})+\sin(\phi_{n-1})]|=2$. It is already well known that in the vicinity of K_c, Eq. (2) shows that the quantities ϕ_n and $\dot{\phi}_n$ present the phenomenon of phase slip, that is, they remain constant for a given period of time T and then they jump, followed by another period of constant value, a jump and so on. Due to the diffusive char-

$$K_c = \frac{|Z_l|}{2} + \frac{|Z_m|}{2} + \epsilon. \tag{5}$$

In order to determine ϵ, we use the equations for the steady state for suffixes $i=l,m$ such that $\dot{\phi}_i = \Delta_i - 2K_c \sin(\phi_i) + K_c \sin(\phi_{i-1}) + K_c \sin(\phi_{i+1}) = 0$. When these two equations are used they will lead exactly to Eqs. (3) and (4), since the sum of the equations of system (1) from 1 to both l and m will give $K_c \sin(\phi_{i+1}) + K_c \sin(\phi_{i-1}) = 2Z_i + 2K_c \sin(\phi_N) - \Delta_i$ for $i=l,m$. Due to the cyclic character of the equations, anything that we do will bring us back to the same equations (3) and (4). At this point we make an educated guess about the quantities that ϵ depends on and then we see how to get its value. In addition to both Z_l and Z_m, the quantities Δ_l and Δ_m appear as parameters in the equations of $\dot{\phi}_l$ and $\dot{\phi}_m$, they are opposite in sign and small under the assumption of large N, large enough so that the initial frequencies of the oscillators are closed packed. Therefore, in such case, taking into consideration that $\epsilon = K_c(\delta/2)$ and the two possibilities $\pm Z_l$ and $\pm\sin(\phi_l)$ associated with $\mp Z_m$ and $\mp\sin(\phi_m)$, we add the two equations for $\dot{\phi}_l$ and $\dot{\phi}_m$ at K_c to obtain

$$K_c \approx \frac{|Z_l| + |Z_m|}{2} + \frac{1}{2}\left\{\frac{|\Delta_l| + |\Delta_m|}{4}\right\}. \tag{6}$$

In the evaluation of this equation, we have used the following approximation for large N: $K_c \sin(\phi_{i+1}) + K_c \sin(\phi_{i-1}) = 2Z_i + 2K_c \sin(\phi_N) - \Delta_i \approx 2Z_i + 2K_c \sin(\phi_N)$ for $i=l,m$. Here we have omitted the Δ_i in Z_i, which are small quantities, thus avoiding to return back to Eqs. (3) and (4), then $Z_i \approx Z_i$ for $i=l,m$. Such approximation will allow us to estimate the parameters that ϵ depends on, since, as we have mentioned, knowing the values of the quantities $|Z_l|$ and $|Z_m|$ will not tell us which pair of oscillators, the one with index l or that with index m, will have the phase-lock condition, unless we perform numerical simulations. Therefore, we arrive to an approximate expression for $\epsilon \approx (|\Delta_l| + |\Delta_m|)/8$. Since the derivation of this quantity is based on an ansatz, it is now necessary to check its validity. When we checked numerically Eq. (6), we found a good fitting for large N with the values of K_c obtained from numerical simulations but for small values of N, the value of K_c calculated from Eq. (6) was larger than that from numerical simulations by a quantity of order 10^{-1}. But, in order to get a more accurate expression for ϵ, we should not forget that we omitted a subtraction of the two quantities Δ_l and Δ_m from both Z_l and Z_m, respectively, in order to arrive at Eq. (6). Therefore, we have to subtract these two quantities Δ_l and Δ_m again from the expression of ϵ considering the two possibilities $\pm Z_l$ and $\pm\sin(\phi_l)$ associated with $\mp Z_m$ and $\mp\sin(\phi_m)$. In order to achieve this, we notice that the quantity ϵ, according to Eq. (6), is related to the average of $|\Delta_l|$ and $|\Delta_m|$, which can also be written in terms of the average between the maximum and minimum of these quantities such that

$$\epsilon \approx \frac{1}{2}\left\{\frac{|\Delta_l| + |\Delta_m| + |\Delta_l + \Delta_m|}{8} + \frac{|\Delta_l| + |\Delta_m| - |\Delta_l + \Delta_m|}{8}\right\}. \tag{7}$$

Thus, we should investigate which quantities are more important to minimize this equation. We checked numerically the calculated value of K_c for the minimum value of Eq. (7) and noticed that K_c depends on the minimum [the second fraction of Eq. (7)] and the accuracy of the calculated one is in good matching with that obtained from numerical simulations within an order of $10^{-2}-10^{-3}$ for small values of N, and it increases as N increases. Thus we obtain an approximate expression for K_c, which we call K_c^a, and is given by

$$K_c^a \approx \frac{|Z_l| + |Z_m|}{2} + \frac{|\Delta_l| + |\Delta_m| - |\Delta_l + \Delta_m|}{16}. \tag{8}$$

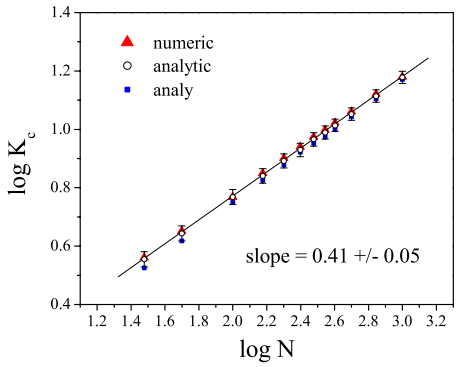

Figure 3 summarizes the numerical simulations as well as the calculations of K_c according to Eq. (8). We plot log K_c versus log N from numerical simulations of Eq. (1) (triangles), from the results given by Eq. (8) (circles), and from Eq. (5) taking ϵ equal to zero (squares). The error bars correspond to the spread of values of K_c for all realizations of the different sets $\{\omega_i\}$ for a given value of N, while the value of K_c plotted is one of these independent realizations. There is no average plotted in this figure. The validity of Eq. (8) is clearly shown for values of N ranging from 30 to 1000. The dependence of K_c on both $|Z_l|$ and $|Z_m|$ as in Eq. (8) and as N increases becomes clear. It can also be inferred that the term which depends on ϵ becomes negligible. This is due to the fact that as N increases the oscillators of indices l and $l+1$ are getting closer in frequencies to each other as well as the two oscillators of indices m and $m+1$. We also observe that for finite N, K_c grows less than $O(\sqrt{N})$, which is found by Strogatz and Mirollo[15] for the coupled oscillators in a chain of free ends. In fact for coupled oscillators in a chain of free ends, the asymptotic behavior is found by considering that the probability of synchronization of N oscillators is related to the maximum excursion of a single pinned random walk (see Ref. 15, Sec. 4), i.e., the probability depends on finding $\max\{|Z_i|\}$ for $i=1,2,3,\ldots,N-1$. In their case such finding is asymptotic and we do not expect it to be exact for finite N.[15] In our case of coupled oscillators in a ring, an equivalent reasoning may complicate matters since K_c depends on the two quantities Z_l and Z_m, where $l<m$, and the probability to find a phase-lock solution is expected to depend on the average of the absolute values of these two quantities, which in principle will lead to the value of K_c growing with a lower exponent than that in the case of chain of free ends, at least

for finite N, as in Fig. 3. This is due to the fact that for finite N, one of the values of either $|Z_l|$ or $|Z_m|$ agrees with $\max\{|Z_i|\}$ for $i=1,2,3,\ldots,N$ and thus the other one will have an absolute value less than $\max\{|Z_i|\}$. Consequently the average value of both of them will be less than $\max\{|Z_i|\}$. Therefore, it is not expected that they will closely follow the case of a chain with free ends for finite values of N, apart from the fact that the result $K_c \sim O(\sqrt{N})$ for the case of free ends is not exact in this case.[15] However, we expect from Eq. (8) that K_c will be of size $O(\sqrt{N})$, as N tends to infinity. The fact that we have these two quantities $|Z_l|$ and $|Z_m|$ may induce us to think that we have two phase differences which will have a saddle node bifurcation at K_c. However, it is still only one pair of oscillators according to Eq. (2) which has the phase-lock condition. This can be verified via numerical simulations since this pair will have a phase slip in the vicinity of K_c. But, we cannot determine which pair, either ϕ_l or ϕ_m, will have the phase-lock condition based only on $|Z_l|$ and $|Z_m|$.

Summarizing, if one knows the set of initial frequencies $\{\omega_i\}$, it is possible to point at the four oscillators at the borders of the major clusters just below K_c and then the calculation of K_c is performed using Eq. (8) (thus obtaining K_c^a) without the need of computer simulation of system (1), just using the values of Z_l and Z_m. If we are interested in determining which phase difference will have a phase-lock condition $\pm\pi/2$, we use the fact that $\sin(\phi_l)$ and $\sin(\phi_m)$ have opposite signs as well as they are maximum and minimum among all values of sine of the phase differences. The sign of the quantity $\sin(\phi_N)$ has the same sign of the quantity $x_1 = -(Z_l+Z_m)/2$, which is taken from the sum of Eqs. (3) and (4) [eliminating for a moment the small difference between $\sin(\phi_l)$ and $\sin(\phi_m)$]. Depending on the signs of Z_l and Z_m, we know the signs of $\sin(\phi_l)$ and $\sin(\phi_m)$, and hence the sign of $\sin(\phi_N)$. Therefore, we count two quantities $x_2 = \pm K_c^a - Z_l$ and $x_3 = \pm K_c^a - Z_m$, positive sign for $Z>0$ and negative sign for the reverse. Three cases will exist: first from the quantities x_2 and x_3, one is positive and the other is negative. Thus depending on the sign of x_1 we choose either x_2 or x_3 to be $K_c^a \sin(\phi_N)$. Second x_2 and x_3 have the same signs, then we check the minimum between $|x_1-x_2|$ and $|x_1-x_3|$ and depending on which one is the minimum, we take either x_2 or x_3 to be $K_c^a \sin(\phi_N)$. Third $|x_1-x_2|=|x_1-x_3|$, then we take the minimum outcome of x_2 and x_3. Now we know the value of $K_c^a \sin(\phi_N)$ and its sign. Therefore, we know which equation (3) or (4) will be used to give K_c^a. Thus we specify which phase difference of index l or m would have $\pm\pi/2$. We tested this method on the simulations we have done and it matched the outcome of the numerical simulations.

The number of oscillators in each cluster at the vicinity of K_c can be determined once we assigned the indices l, $l+1$, m, and $m+1$, in which we remind the reader, obtained from Z_l and Z_m, maximum and minimum values of the sequence Z_i.The size of one cluster of N_1 oscillators is determined by counting the difference $N_1=(m+1)-l$ and the size of the other cluster is determined as $N_2=N-N_1$. Similar to the calculation of Z_l and Z_m, we can determine other two quantities $Y_1=N_1\omega_0-\Sigma_{i=l+1}^{m}\omega_i$ and $Y_2=N_2\omega_0-\Sigma_{i=m+1}^{l}\omega_i$, taking into consideration the periodic boundary conditions. It is found that $|Y_1|=|Y_2|$. These quantities are related to Z_l and Z_m by $Y_1=Z_m-Z_l=-Y_2$. It is easy to show that $K_c=|Y_1|/2+\epsilon = |Y_2|/2+\epsilon$. The two quantities Y_1 and Y_2 provide a criterion to understand synchronization-desynchronization at K_c. If one arrives from above K_c where all oscillators are synchronized and have the same value of frequency, at K_c the oscillators split into two groups of N_1 and N_2 at K_c, depending on these two quantities Y_1 and Y_2, where $|Y_1|=|Y_2|$. It is not necessary for N_1 to be equal to N_2. Both quantities Y_1 and Y_2 have opposite signs since they refer to two groups of oscillators (two clusters): one of them rotates with average frequency above ω_0 and the other has an average frequency lower than ω_0.

Comparing our findings of K_c with the work of Daniels *et al.*,[10] our method has the advantage of finding the value of K_c without performing numerical simulations once we know the set of initial frequencies $\{\omega_i\}$. In addition we get the condition of synchronization-desynchronization at K_c and obtain the number of oscillators in each branch in the vicinity of K_c.

III. CONCLUSION

We have analyzed the conditions of the phase differences for the onset of complete synchronization at the critical coupling strength in a Kuramoto-like model with nearest neighbor coupling with periodic boundary conditions. Such analysis allows us to determine the four oscillators located at the borders of the major clusters (formed by successive oscillator) which will meet at the critical coupling to form one cluster of all synchronized oscillators. With the help of these findings and a justified educated guess we derive a mathematical expression for the critical coupling when all oscillators will have the same frequency and phase differences and instantaneous velocities become time independent. In addition, we are able to determine which is the phase difference, which will have a phase-lock solution $\pm\pi/2$. From the derivation we also extract the size of the clusters before complete synchronization. The expression for K_c depends only on the initial frequencies, through the quantities Z_l and Z_m, where the indices l and m correspond to the borders of the clusters. The quantities Z_l and Z_m correspond to the maximum and minimum values of the sequence Z_i. These quantities in fact are related to the statistics of the distribution of the set of initial frequencies $\{\omega_i\}$, when this sample is obtained from a Gaussian distribution, as shown by Strogatz and Mirollo,[15] for the case of a chain with free ends. A detailed study within this context could shine light on the behavior of K_c for finite N as well as $N\to\infty$, not just for the case of a Gaussian distribution but for others. It will be interesting to investigate the relationship that the quantities Z_l and Z_m and/or the quantities Y_1 and Y_2 have with the statistical behavior of the system of periodic boundary conditions, and how this approaches the case of free ends, but it is outside the scope of this study. This investigation plus extension of the method to study cluster formation inside the tree will be topics of further analysis. The advantages of the study presented here are that we can determine the value of the coupling constant that will synchronize the system of coupled oscillators without carrying out numerical simula-

tion as well as to determine the sizes of the clusters just before this happens. Generalization of these results to different couplings and boundaries is under investigation and will be presented elsewhere.

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